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On the stability of the boundary element collocation method applied to the linear heat equation

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Abstract

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The boundary element method (boundary integral equation method) is considered for the Dirichlet problem of the heat equation. The method of collocations on the boundary using finite-element basis is applied to the discretization of the Volterra integral equation of the first kind. Ill-posed properties of the coefficient matrices are discussed.

Keywords: Dirichlet problem, boundary integal equation method, finite-element collocation, Volterra integral equation of the first kind, ill-posedness.

1. Introduction

Recently, numerical solutions of the transient heat conduction problems are often obtained by the boundary element method. This method has been preferred to domain-type methods in engineering, particularly for the exterior problems, because the problems can be treated in the same way as the interior problems. When the time-dependent fundamental solution is used in the formulation, our aimed equation defined on the boundary is an integral equation of Volterra type.

For the Neumann problem, many mathematical results are given by several authors. Iso et al. [3] showed the uniform convergence for the boundary element collocation method for the case of convex and smooth boundaries. Costabel et al. [2] showed the existence of the continuous solution of the integral equation on a nonsmooth boundary. Onishi [6] presented the projection method.

The discussions on the Dirichlet problem are rather new. The corresponding boundary equation is the Volterra integral equation of the first kind. Okamoto [5] discussed the stability and convergence for the boundary element Galerkin method. Noon [4] presented the Galerkin method for the solution in the space $H^{-1/2,-1/4}$. Onishi [7] noted the ill-posedness of the approximation. More recently, Costabel [1] showed the Hilbert theory for the integral equation and its approximation.

In this paper, we consider the Dirichlet problem for the case of smooth boundaries. The Volterra integral equation of the first kind is approximated by the method of boundary element collocations. We shall examine the norm of the coefficient matrices and present the ill-posedness of the approximation.

2. Boundary integral equation

We consider the Dirichlet problem of the linear heat equation for an unknown temperature $u(x, t)$:

$$\frac{\partial u}{\partial t} = \Delta u, \qquad \text{in } Q = \Omega \times (0, T], \tag{1}$$

$$u(x, 0) = u^0(x), \qquad \text{on } \bar{\Omega}, \tag{2}$$

$$u(x, t) = \bar{u}(x, t), \qquad \text{on } \Sigma = \Gamma \times [0, T], \tag{3}$$

where Ω is a bounded domain in d-dimensional Euclidean space $\mathbb{R}^d$ ($d = 2$ or 3), of which the boundary Γ is smooth, and where Δ denotes the Laplacian in $\mathbb{R}^d$. We assume that the Cauchy data u^0 and the Dirichlet data $\bar{u}$ are smooth enough and satisfy some compatibility conditions, so that the above problem has a unique solution u in $C^2(\bar{Q})$.

The solution $u(x, t)$ at any point $x \in \Omega$ can be represented in the following integral form:

$$\begin{aligned} u(x, t) = &-\int_0^t \int_\Gamma \frac{\partial v}{\partial n(y)}(y, \tau; x, t) \bar{u}(y, \tau) \, \mathrm{d}\Gamma(y) \, \mathrm{d}\tau \\ &+ \int_0^t \int_\Gamma v(y, \tau; x, t) q(y, \tau) \, \mathrm{d}\Gamma(y) \, \mathrm{d}\tau \\ &+ \int_\Omega v(y, 0; x, t) u^0(y) \, \mathrm{d}\Omega(y), \quad 0 < t \leqslant T, \end{aligned} \tag{4}$$

where $v(y, \tau; x, t)$ denotes the fundamental solution of the adjoint heat operator $\partial/\partial\tau + \Delta_y$ in $\mathbb{R}^d \times (0, \infty)$, which is presented by

$$v(y, \tau; x, t) = \left(\frac{1}{\sqrt{4\pi(t-\tau)}} \right)^d \exp\left(-\frac{r^2}{4(t-\tau)} \right), \tag{5}$$

with $r = |x - y|$, and where $q(y, \tau) = \partial u(y, \tau)/\partial n(y)$ is the heat flux along the exterior normal $n(y)$ to the boundary Γ.

By letting a point x tend to the boundary in the nontangential direction in (4), we have the boundary integral equation

$$\tfrac{1}{2}\bar{u}(x,\,t) + \text{p.v.}\int_0^t\int_\Gamma \frac{\partial v}{\partial n(y)}(y,\,\tau;\,x,\,t)\bar{u}(y,\,\tau)\,\mathrm{d}\Gamma(y)\,\mathrm{d}\tau$$
$$= \int_0^t\int_\Gamma v(y,\,\tau;\,x,\,t)q(y,\,\tau)\,\mathrm{d}\Gamma(y)\,\mathrm{d}\tau + \int_\Omega v(y,0;\,x,\,t)u^0(y)\,\mathrm{d}\Omega(y), \tag{6}$$

where p.v. indicates Cauchy's principal values of integrals. This integral equation is derived from the interior problem, and we must remark that the same equation can be derived from the exterior problem. The heat flux $q(x,\,t)$ is unknown involved in the integral equation (6), and we abbreviate the equation (6) as the following Volterra integral equation:

$$\int_0^t\int_\Gamma v(y,\,\tau;\,x,\,t)q(y,\,\tau)\,\mathrm{d}\Gamma(y)\,\mathrm{d}\tau = g(x,\,t), \quad \text{on } \Sigma, \tag{7}$$

with the right-hand side

$$g(x,\,t) := \tfrac{1}{2}\bar{u}(x,\,t) + \text{p.v.}\int_0^t\int_\Gamma \frac{\partial v}{\partial n(y)}(y,\,\tau;\,x,\,t)\bar{u}(y,\,\tau)\,\mathrm{d}\Gamma(y)\,\mathrm{d}\tau$$
$$- \int_\Omega v(y,0;\,x,\,t)u^0(y)\,\mathrm{d}\Omega(y). \tag{8}$$

We note that the integral on the left-hand side of (7) has a weak singularity at $x=y$ as $\tau \to t-0$. In fact

$$v = \frac{4^{d/2-\mu}}{2^d\pi^{d/2}}\frac{1}{(t-\tau)^\mu}\frac{1}{r^{d-2\mu}}\left(\frac{r^2}{4(t-\tau)}\right)^{d/2-\mu}\exp\left(-\frac{r^2}{4(t-\tau)}\right),$$

for any $\mu > 0$, and we can see, from the inequality $\xi^s\mathrm{e}^{-\xi} \leqslant s^s\mathrm{e}^{-s}$, $\xi > 0$, that

$$|v| \leqslant \frac{1}{4^\mu\pi^{d/2}}\frac{1}{(t-\tau)^\mu}\frac{1}{r^{d-2\mu}}s^s\,\mathrm{e}^{-s}, \tag{9}$$

for $s = \frac{1}{2}d - \mu > 0$. Accordingly, the integral is absolutely convergent for $\frac{1}{2} < \mu < 1$.

The existence and uniqueness of the solution of the integral equation (7) with its boundary regularity property on Lipschitz cylinders is discussed in [1]. Here we assume that the integral equation has a unique solution q, which is as much smooth as we please by implicitly assuming the sufficient smoothness and compatibility in both geometry and data.

We remark finally the singularity of the integral kernel of the right-hand side of (8) on the Liapunov boundary. The boundary Γ is called the Liapunov boundary with the index κ, $0 < \kappa \leqslant 1$, when there exists a constant $L(\Gamma) > 0$, depending only on Γ, such that

$$|\cos\vartheta| \leqslant L\,|x-y|^\kappa, \quad \text{on } \Gamma,$$

where ϑ is the angle between the normal $n(y)$ to Γ and the positional vector $x-y$. When $\kappa = 1$, the curvature of the boundary is continuous. Since the normal derivative is expressed in the form

$$\frac{\partial v}{\partial n(y)} = \frac{1}{2}\left(\frac{1}{\sqrt{4\pi(t-\tau)}}\right)^d\frac{r}{t-\tau}\exp\left(-\frac{r^2}{4(t-\tau)}\right)\cos\vartheta,$$

and

$$\frac{\partial v}{\partial n}=\frac{4^{1+d/2-\nu}}{2^{1+d}\pi^{d/2}}\frac{1}{(t-\tau)^{\nu}}\frac{\cos\vartheta}{r^{1+d-2\nu}}\left(\frac{r^2}{4(t-\tau)}\right)^{1+d/2-\nu}\exp\left(-\frac{r^2}{4(t-\tau)}\right),$$

for any $\nu>0$, we can see that

$$\left|\frac{\partial v}{\partial n}\right|\leqslant\frac{L}{2^{2\nu-1}\pi^{d/2}}\frac{1}{(t-\tau)^{\nu}}\frac{1}{r^{1+d-2\nu-\kappa}}s^s\,\mathrm{e}^{-s},\tag{10}$$

for $s=1+\frac{1}{2}d-\nu>0$. Accordingly, the integral is absolutely convergent for $1-\frac{1}{2}\kappa<\nu<1$.

3. Method of boundary element collocations

We adopt the method of collocations using the finite-element basis for approximating the boundary integral equation (7). To this end, we divide Γ into small curved simplexes (curved triangles for $d=3$, arc segments for $d=2$). In this paper, we assume that the totality of these curved simplexes coincides with Γ, and there occur no discrepancies. We call each of these simplexes a boundary element.

Let $\{\phi_j(y)\}_{j=1}^n$ be a set of the finite-element basis defined on the boundary elements, and let h denote a representative diameter in refinement. Then we approximate the exact solution $q(y,\tau)$ of the equation (7) in the form

$$q_h(y,\tau)=\sum_{j=1}^{n}q_j(\tau)\phi_j(y).\tag{11}$$

Let $\{x_i\}_{i=1}^n$ denote a set of collocation points. Since the unknown function q_j in (11) should be determined by the equations

$$\int_0^t \mathrm{d}\tau\int_\Gamma v(y,\tau;\,x_i,\,t)q_h(y,\tau)\,\mathrm{d}\Gamma(y)=g(x_i,t),\quad i=1,2,\ldots,n,\tag{12}$$

substitution of (11) into (12) yields the following system of Volterra integral equations for unknowns $q_j(\tau)$:

$$\sum_{j=1}^{n}\int_0^t V_{ij}(\tau,t)q_j(\tau)\,\mathrm{d}\tau=g(x_i,t),\quad i=1,2,\ldots,n,\tag{13}$$

where

$$V_{ij}(\tau,t):=\int_\Gamma v(y,\tau;\,x_i,\,t)\phi_j(y)\,\mathrm{d}\Gamma(y).\tag{14}$$

To solve (13) approximately, we divide the time interval $[0,T]$. Let Δt be defined by $\Delta t=T/N$, and set $t_k:=k\,\Delta t$, $k=0,1,\ldots,N$. And let $\{\psi_k(t)\}_{k=0}^N$ be a set of the basis corresponding to the subdivision. In this paper, we assume that ψ_k is a piecewise-linear function or a step function. Then the coefficient $q_j(\tau)$ is sought in the form

$$q_j^{\Delta t}(\tau)=\sum_{k=0}^{m}q_j^k\psi_k(\tau),\quad 0\leqslant\tau\leqslant t_m,\tag{15}$$

with their initial values $q_j^0 = \partial u^0(x_j)/\partial n$. The coefficients q_j^k stand for the approximate values to $q(x_j, t_k)$. They are determined by solving the following equations successively:

$$\sum_{j=1}^{n} \int_0^{t_m} V_{ij}(\tau, t_m) q_j^{\Delta t}(\tau)\, \mathrm{d}\tau = g(x_i, t_m), \quad i = 1, 2, \ldots, n, \quad m = 1, 2, \ldots, N. \tag{16}$$

Therefore, by substitution of (15) into (16), this turns out to be a recurrence system of linear-algebraic equations with respect to m:

$$\sum_{j=1}^{n} \tilde{a}_{ij}^{(m)} q_j^m = g(x_i, t_m) - \sum_{k=0}^{m-1} \sum_{j=1}^{n} a_{ij}^{(k)} q_j^k, \quad i = 1, 2, \ldots, n, \tag{17}$$

with the coefficients

$$a_{ij}^{(k)} := \int_0^{t_m} V_{ij}(\tau, t_m) \psi_k(\tau)\, \mathrm{d}\tau, \quad 0 \leqslant k \leqslant m-1, \ 1 \leqslant i, \ j \leqslant n, \tag{18}$$

$$\tilde{a}_{ij}^{(m)} := \int_0^{t_m} V_{ij}(\tau, t_m) \psi_m(\tau)\, \mathrm{d}\tau, \quad 1 \leqslant i, \ j \leqslant n. \tag{19}$$

Introducing an n-square matrix $[A^{(k)}]$ (respectively $[\tilde{A}^{(m)}]$) with its ij-element $a_{ij}^{(k)}$ (respectively $\tilde{a}_{ij}^{(m)}$), an n-column vector $\{q^{(k)}\}$ with its jth component q_j^m, and $\{g^{(m)}\}$ with its ith component $g(x_i, t_m)$, we can write (17) in the form

$$[\tilde{A}^{(m)}]\{q^{(m)}\} = \{g^{(m)}\} - \sum_{k=0}^{m-1} [A^{(k)}]\{q^{(k)}\}.$$

When $[\tilde{A}^{(m)}]$ is invertible, the nodal values q_j^m are given by

$$\{q^{(m)}\} = [\tilde{A}^{(m)}]^{-1} \left(\{g^{(m)}\} - \sum_{k=0}^{m-1} [A^{(k)}]\{q^{(k)}\} \right), \quad m = 1, 2, \ldots, N. \tag{20}$$

In practice of computation, the Dirichlet data $\bar{u}$ on the right-hand side of (8) are often discretized in the form

$$\bar{u}^{\mathrm{I}}(y, \tau) = \sum_{j=0}^{n} \sum_{k=0}^{m} \bar{u}(x_j, t_k) \psi_k(\tau) \phi_j(y), \quad 0 \leqslant \tau \leqslant t_m. \tag{21}$$

Substitution of (21) into (8) yields

$$g^{\mathrm{I}}(x_i, t_m) := \tfrac{1}{2}\bar{u}(x_i, t_m) + \sum_{k=0}^{m} \sum_{j=1}^{n} b_{ij}^{(k)} \bar{u}(x_j, t_k)$$
$$- \int_\Omega v(y, 0;\, x_i, t_m) u^0(y)\, \mathrm{d}\Omega(y), \quad i = 1, 2, \ldots, n,$$

with the coefficients

$$b_{ij}^{(k)} := \int_0^{t_m} \psi_k(\tau) \int_\Gamma \frac{\partial v}{\partial n(y)}(y, \tau;\, x_i, t_m) \phi_j(y)\, \mathrm{d}\Gamma(y). \tag{22}$$

We note that $g(x_i, t_m)$ in (17) is replaced by $g^{\mathrm{I}}(x_i, t_m)$ in practical computation. The n-square matrix with its ij-element $b_{ij}^{(k)}$ is denoted by $[B^{(k)}]$.

In numerical practice in engineering, the piecewise-linear finite-element bases are often used as $\{\phi_j(y)\}_{j=1}^n$, while either the piecewise constant ($\sigma = 1$) or the linear ($\sigma = 2$) finite-element bases are used as $\{\psi_k(\tau)\}_{k=0}^N$. Finally we remark that for the former case $[\tilde{A}^{(m)}] = [A^{(m)}]$ and $[A^{(0)}] = [O]$, but that $[\tilde{A}^{(m)}] \neq [A^{(m)}]$ for the latter case.

4. Matrix properties

We show ill-posed properties of the boundary element approximation to the Volterra integral equation of the first kind. We focus on the ill-posedness in the numerical approximation. To this end, we fix our idea first to the linear elements in both space and time.

Theorem 1. *For the linear basis $\{\phi_j(y)\}_{j=1}^n$ and $\{\psi_k(\tau)\}_{k=0}^N$, there exists a positive constant $G_\mu(\Gamma)$, independent of h or Δt, such that*

$$\| [\tilde{A}^{(m)}] \|_\infty \leqslant G_\mu(\Gamma) \frac{\Delta t^{1-\mu}}{(1-\mu)(2-\mu)}, \tag{23}$$

$$\| [A^{(k)}] \|_\infty \leqslant G_\mu(\Gamma) \frac{2(2^{1-\mu}-1)}{(1-\mu)(2-\mu)}\, \Delta t^{1-\mu}, \quad k = 0, 1, 2, \ldots, m-1, \tag{24}$$

for any μ, $\frac{1}{2} < \mu < 1$.

Proof. Prior to the proof, we remark that $\psi_k(\tau)$ is a piecewise-linear function which satisfies $\psi_k(t_m) = \delta_{k,m}$, and that $\Sigma_{j=1}^n \phi_j(x) \equiv 1$. We show (23) first. Since $\tilde{a}_{ij}^{(m)} > 0$, we can see, from (9) and (14), that

$$\begin{aligned}\sum_{j=1}^n |\tilde{a}_{ij}^{(m)}| &= \int_{t_{m-1}}^{t_m} \psi_m(\tau) \int_\Gamma v(y, \tau;\, x_i, t_m)\, \mathrm{d}\Gamma(y)\, \mathrm{d}\tau \\ &\leqslant \frac{s^s\, \mathrm{e}^{-s}}{4^\mu \pi^{d/2}} \int_{t_{m-1}}^{t_m} \frac{\psi_m(\tau)}{(t_m-\tau)^\mu}\, \mathrm{d}\tau \int_\Gamma \frac{\mathrm{d}\Gamma(y)}{|x_i - y|^{d-2\mu}},\end{aligned}$$

with $s = \frac{1}{2}d - \mu$. By the change of variable, $\tau \mapsto t_{m-1} + \theta\, \Delta t$, we have

$$\int_{t_{m-1}}^{t_m} \frac{\psi_m(\tau)}{(t_m-\tau)^\mu}\, \mathrm{d}\tau = \Delta t^{1-\mu} \int_0^1 \frac{\theta\, \mathrm{d}\theta}{(1-\theta)^\mu} = \frac{\Delta t^{1-\mu}}{(1-\mu)(2-\mu)}.$$

Put

$$G_\mu(\Gamma) := \frac{s^s\, \mathrm{e}^{-s}}{4^\mu \pi^{d/2}} \max_{x \in \Gamma} \int_\Gamma \frac{\mathrm{d}\Gamma(y)}{|x-y|^{d-2\mu}};$$

then $0 < G_\mu < \infty$ for $\frac{1}{2} < \mu < 1$, hence we get (23).

Similarly as before, we can see

$$\begin{aligned}\sum_{j=1}^n |a_{ij}^{(k)}| &= \int_{t_{k-1}}^{t_{k+1}} \psi_k(\tau) \int_\Gamma v(y, \tau;\, x_i, t_m)\, \mathrm{d}\Gamma(y)\, \mathrm{d}\tau \\ &\leqslant \frac{s^s\, \mathrm{e}^{-s}}{4^\mu \pi^{d/2}} \int_{t_{k-1}}^{t_{k+1}} \frac{\psi_k(\tau)}{(t_m-\tau)^\mu}\, \mathrm{d}\tau \int_\Gamma \frac{\mathrm{d}\Gamma(y)}{|x_i - y|^{d-2\mu}}.\end{aligned}$$

Here we get

$$\int_{t_{k-1}}^{t_{k+1}} \frac{\psi_k(\tau)}{(t_m-\tau)^{\mu}}\,\mathrm{d}\tau = \frac{\Delta t^{1-\mu}}{(1-\mu)(2-\mu)} \times\left\{(m-k+1)^{2-\mu} - 2(m-k)^{2-\mu} + (m-k-1)^{2-\mu}\right\} \leqslant 2(2^{1-\mu}-1)\,\Delta t^{1-\mu}.$$

This completes the proof. □

Especially we are interested in the limiting case of (23) and (24) as $\mu \to \frac{1}{2}$. For the case of linear elements in space with $d=2$ and constant elements in time, we have the following estimate.

Theorem 2. *For the linear basis $\{\phi_j(y)\}_{j=1}^{n}$ with $d=2$ and the constant basis $\{\psi_k(\tau)\}_{k=0}^{N}$, there exist two positive constants c_1 and c_2, independent of h or Δt, such that*

$$c_1\sqrt{\Delta t} \leqslant \|[A^{(m)}]\|_{\infty} \leqslant c_2\sqrt{\Delta t}. \tag{25}$$

Proof. As in the proof of Theorem 1, we estimate

$$\sum_{j=1}^{n} \left|a_{ij}^{(m)}\right|.$$

However we estimate it, here, not by using (9) but by direct calculation, since we can calculate it by introduction of a coordinate system to Γ. Hence we get the estimate of the theorem. □

Corollary. *Under the same assumption as in Theorem 2, if $[A^{(m)}]$ is regular, then we have*

$$\frac{1}{c_2\sqrt{\Delta t}} \leqslant \|[A^{(m)}]^{-1}\|_{\infty}. \tag{26}$$

From this result, we know that the recurrence relation (20) is ill-posed and the solution is sensitive to any disturbances in the column vector $\{g^{(m)}\}$.

Next we consider the Dirichlet problem in one dimension: $d=1$, $\Omega=(a, b)$. In this case, the coefficient matrix $[\tilde{A}^{(m)}]$ is given by

$$[\tilde{A}^{(m)}] = \begin{pmatrix} -G_1 & G_2 \\ -G_2 & G_1 \end{pmatrix},$$

where

$$G_1 := \int_{t_{m-1}}^{t_m} \psi_m(\tau) v(a, \tau; a, t_m)\,\mathrm{d}\tau,$$

$$G_2 := \int_{t_{m-1}}^{t_m} \psi_m(\tau) v(b, \tau; a, t_m)\,\mathrm{d}\tau.$$

The inverse of the matrix and its maximum norm are explicitly given as follows:

$$[\tilde{A}^{(m)}]^{-1} = \frac{1}{G_2^2 - G_1^2} \begin{pmatrix} G_1 & -G_2 \\ G_2 & -G_1 \end{pmatrix},$$

$$\| [\tilde{A}^{(m)}]^{-1} \|_\infty = \frac{1}{G_1 - G_2}.$$

For the linear basis $\{\psi_k(t)\}_{k=1}^N$, we have

$$G_1 = \frac{2}{3\sqrt{\pi}} \sqrt{\Delta t},$$

$$G_1 - G_2 = \frac{\sqrt{\Delta t}}{2\sqrt{\pi}} \int_0^1 \frac{1-\theta}{\sqrt{\theta}} \left(1 - \exp\left(-\frac{|b-a|^2}{4\,\Delta t \theta}\right)\right) \mathrm{d}\theta$$

$$> \frac{\sqrt{\Delta t}}{2\sqrt{\pi}} \int_0^1 \frac{1-\theta}{\sqrt{\theta}} \,\mathrm{d}\theta \left(1 - \exp\left(-\frac{|b-a|^2}{4\,\Delta t}\right)\right). \tag{27}$$

Accordingly we know

$$\frac{3\sqrt{\pi}}{2\sqrt{\Delta t}} = \frac{1}{G_1} < \| [A^{(m)}]^{-1} \|_\infty$$

$$< \frac{3\sqrt{\pi}}{2\sqrt{\Delta t}} \frac{1}{1 - \exp(-|b-a|^2/(4\,\Delta t))}. \tag{28}$$

This indicates that $\|[A^{(m)}]^{-1}\|_\infty = \mathrm{O}(1/\sqrt{\Delta t})$ as $\Delta t \to 0$. The estimate (27) implies that the matrix $[\tilde{A}^m]$ is invertible for any Δt.

Incidentally, for the constant basis, we have the following result:

$$\frac{\sqrt{\pi}}{\sqrt{\Delta t}} < \| [A^{(m)}]^{-1} \|_\infty < \frac{\sqrt{\pi}}{\sqrt{\Delta t}} \frac{1}{1 - \exp(-|b-a|^2/(4\,\Delta t))}. \tag{29}$$

From (26), (28) and (29), we might conjecture that

$$\frac{\alpha}{\sqrt{\Delta t}} \leqslant \| [\tilde{A}^{(m)}]^{-1} \|_\infty \leqslant \frac{\beta}{\sqrt{\Delta t}}, \tag{30}$$

with constants $\beta > \alpha > 0$ for $d = 3$.

These results indicate necessity of some regularization for the direct boundary element collocation method to the Dirichlet problem of the heat equation.

Finally, we add some estimates of the matrix $[B^k]$.

Theorem 3. *For the linear basis* $\{\psi_j(y)\}_{j=1}^n$ *and* $\{\psi_k(\tau)\}_{k=0}^N$, *there exists a positive constant* $H_\nu(\Gamma)$, *independent of* h *or* Δt, *such that*

$$\| [B^{(m)}] \|_\infty \leqslant H_\nu(\Gamma) \frac{\Delta t^{1-\nu}}{(1-\nu)(2-\nu)}, \tag{31}$$

$$\| [B^{(k)}] \|_\infty \leqslant H_\nu(\Gamma) \frac{2(2^{1-\nu}-1)}{(1-\nu)(2-\nu)} \Delta t^{1-\nu}, \quad k = 0, 1, 2, \ldots, m-1, \tag{32}$$

for any ν, $1 - \frac{1}{2}\kappa < \nu < 1$.

Proof. We use the same technique as in Theorem 1. We can see from (22) that

$$\sum_{j=1}^{n} \left| b_{ij}^{(m)} \right| = \int_{t_{m-1}}^{t_m} \psi_m(\tau) \int_\Gamma \left| \frac{\partial v}{\partial n(y)}(y, \tau; x_i, t_m) \right| \mathrm{d}\Gamma(y) \, \mathrm{d}\tau$$

$$\leqslant \frac{Ls^s \, \mathrm{e}^{-s}}{2^{2\nu-1}\pi^{d/2}} \int_{t_{m-1}}^{t_m} \frac{\psi_m(\tau)}{(t_m - \tau)^\nu} \, \mathrm{d}\tau \int_\Gamma \frac{\mathrm{d}\Gamma(y)}{|x_i - y|^{1+d-2\nu-\kappa}}$$

$$\leqslant H_\nu(\Gamma) \frac{\Delta t^{1-\nu}}{(1-\nu)(2-\nu)},$$

with the constant

$$H_\nu(\Gamma) := \frac{Ls^s \, \mathrm{e}^{-s}}{2^{2\nu-1}\pi^{d/2}} \max_{x \in \Gamma} \int_\Gamma \frac{\mathrm{d}\Gamma(y)}{|x - y|^{1+d-2\nu-\kappa}}.$$

Hence we get (31). The inequality (32) can be proved in the similar way. □

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